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AFCRC-TN-59-103
ASTIA DOCUMENT NO. AD 208235

NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

RESEARCH REPORT No. EM-121

Diffraction by an Elliptic Cylinder

BERTRAM LEVY

Contract No. AF 19(604)1717

DECEMBER, 1958

EM-121
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DIFFRACTION BY AN ELLIPTIC CYLINDER

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The research reported in this document has been sponsored by the Electronics Research Directorate of the Air Force Cambridge Research Center, Air Research and Development Command. The publication of this report does not necessarily constitute approval by the Air Force of the findings or conclusions contained herein.

Contract No. AF 19(604)1717

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(31) $V_n^{(1)\prime}(a) \sim \pi^{-1} \zeta_n^{1/4} 3^{2/3} (2\tau_n^*)^{-1/4} k^{5/6} (\sinh a)^{-1/6} (\cosh a)^{-1/6} e^{-i\pi/3} A'(q_n) \frac{\partial\zeta_n}{\partial\xi}$

(32) $\frac{\partial}{\partial b} V_n^{(1)}(a) \sim \pi^{-1} \zeta_n^{1/4} 3^{2/3} (2\tau_n^*)^{-1/4} k^{5/6} (\sinh a)^{-1/6} (\cosh a)^{-1/6} e^{-i\pi/3} A'(q_n) \frac{\partial\zeta_n}{\partial b}$

(33) $\frac{\partial\zeta_n}{\partial\xi} \sim \zeta_n^{-1/2} (2\tau_n^*)^{1/2} k^{-1/3} (\sinh a)^{1/3} (\cosh a)^{1/3}$

(36) $u(\xi, \eta) \sim e^{i\pi/3} k^{-2/3} \pi (\sinh a)^{1/6} (\cosh a)^{2/3} 12^{-1} 6^{-1/3} \ldots$

Abstract

The boundary value problem corresponding to a cylindrical wave incident upon an acoustically soft or acoustically hard elliptic cylinder is formulated and solved by the usual separation of variables technique. This solution is then asymptotically expanded for large values of $kh = 2\pi h/\lambda$, where λ is the wavelength, and h is one half the interfocal distance of the generating ellipse of the cylinder. The short wave length expansion of the field is also obtained by applying the newly developed geometrical diffraction theory, and compared with the asymptotic expansion of the diffracted field. For both the soft and hard cylinders it is shown that the geometrical solution is the leading term of the asymptotic expansion of the exact solution.

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1. Introduction

The construction of the field diffracted by a cylinder of arbitrary convex cross section has been carried out by J.B. Keller[1], who found that by introducing a new class of diffracted rays a "geometrical optics" for diffracted fields could be developed. In all cases which have been hitherto studied it has been found that the field constructed by the geometrical theory of diffraction agrees with an asymptotic expansion of the exact solution to the corresponding boundarv value problem. In this paper we find the asymptotic expansion of the field diffracted by an elliptic cylinder and show that it, too, is in agreement with the geometrical theory.

In Section 2 we formulate the boundary value problem for the reduced wave equation corresponding to a cylindrical wave incident upon an acoustically soft or acoustically hard elliptic cylinder. The problem is then solved by the usual separation of variables technique, and an alternative representation for the field is presented. By using the asymptotic expansions for the special functions occurring in the solution the asymptotic expansion of the diffracted field is found in Section 3. In Section 4 the diffracted field is constructed geometrically and this construction is shown to be in agreement with the asymptotic expansion of the exact solution.

In the first appendix some identities which are necessary to affect the comparison of the geometric construction and the asymptotic expansion of the exact solution are presented. In the second appendix the asymptotic expansions used in Section 3 are derived. The third appendix contains a rigorous justification of the alternative representation of the field given in Section 2.

2. The exact solution

Consider an elliptic cylinder with its axis coincident with the z axis of an (x,y,z) rectangular coordinate system. Let the interfocal distance of the generating ellipse be 2h and let us introduce elliptic coordinates in the (x,y) plane by

$$x = h \cosh \xi \cos \eta \tag{1}$$

$$y = h \sinh \xi \sin \eta \,. \tag{2}$$

We assume that the generating ellipse is given by $\xi = a$ and that a line source is located on the x axis at $\eta = 0$, $\xi = \xi_o$. If the propagation constant of the medium exterior to the cylinder is k_1 then the diffraction problem we will solve is the two-dimensional problem of finding a solution $u(\xi,\eta)$ of

$$\left(\nabla^2 + k_1^2\right)u = \delta\left(\xi - \xi_o\right)\delta(\eta) \tag{3}$$

which satisfies the Sommerfeld radiation condition and prescribed boundary conditions on the surface of the cylinder

$$u(a,\eta) = 0 \qquad \text{(acoustically soft cylinder)} \tag{4}$$

or

$$u_\xi(a,\eta) = 0 \qquad \text{(acoustically hard cylinder).} \tag{5}$$

When we write equation (3) in elliptic coordinates it becomes

$$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + 2\kappa^2(\cosh 2\xi - \cos 2\eta)u = \delta\left(\xi - \xi_o\right)\delta(\eta) \tag{6}$$

with

$$2\kappa = k_1 h. \tag{7}$$

If we replace the delta functions on the right-hand side of equation (6) by zero, then solutions of (6) in the form $\phi(\eta)\Psi(\xi)$ exist where ϕ and Ψ satisfy the equations

$$\frac{d^2\phi}{d\eta^2} + k^2(b^2 - \cos^2\eta)\phi = 0 \tag{8}$$

$$\frac{d^2\Psi}{d\xi^2} - k^2(b^2 - \cosh^2\xi)\Psi = 0, \tag{9}$$

with

$$k = k_1 h \tag{10}$$

and b an arbitrary separation constant.

In order to solve equation (6) by the usual separation of variables technique we observe that the solution must be 2π periodic in η. Thus in the usual series representation of the solution the solutions of (8) which appear must be periodic in η. Let us define solutions $c(\eta,b)$ and $s(\eta,b)$ of (8), which satisfy the initial conditions

$$\begin{aligned} c(o,b) &= 1 & s(o,b) &= 0 \\ c'(o,b) &= 0 & s'(o,b) &= 1. \end{aligned} \tag{11}$$

Then for a discrete set of values b_n of b, the functions c and s which satisfy (11) and (8) will be 2π periodic in η. An even and an odd periodic solution of (8) cannot coexist for the same value of b_n. However, we will use, almost exclusively, only the even solution of the equation and hence no confusion will arise. For these values of b let us denote the corresponding functions by $c_n(\eta)$ and $s_n(\eta)$. Notice that $c_n(\eta)$ and $s_n(\eta)$ are multiples of the standard Mathieu functions $ce_n(\eta)$ and $se_n(\eta)$. In fact,

$$ce_n(\eta) = \sqrt{\pi}\, c_n(\eta) \left[\int_o^{2\pi} c_n^2(\eta)\, d\eta \right]^{-\frac{1}{2}} \tag{12}$$

and

$$(13) \qquad se_n(\eta) = \sqrt{\pi}\, s_n(\eta) \left[\int_0^{2\pi} s_n^2(\eta)\, d\eta \right]^{-\frac{1}{2}} .$$

The solutions of (9) are characterized by their asymptotic behavior as $\xi \to \infty$. The solution having the character of an outgoing wave is denoted by $v^{(1)}(\xi)$, while the incoming solution is denoted by $v^{(2)}(\xi)$. At the values b_n the corresponding functions are denoted by $v_n^{(1)}(\xi)$ and $v_n^{(2)}(\xi)$, which are multiples of the standard Mathieu functions $Me_n^{(1)}(\xi)$ and $Me_n^{(2)}(\xi)$.

To solve (6) we make use of the completeness of the angular Mathieu functions and the fact that the solution is even in η to write u as

$$(14) \qquad u(\xi,\eta) = \sum_{n=0}^{\infty} v_n(\xi) c_n(\eta).$$

Upon substituting (14) into (6) and making use of the orthogonality property of the c_n functions we find that $v_n(\xi)$ satisfies the equation

$$(15) \qquad v_n'' - k^2(b_n^2 - \cosh^2\xi)v_n = \delta(\xi-\xi_o)\left[\int_0^{2\pi} c_n^2(\eta)\, d\eta \right]^{-1} .$$

A unique solution of equation (15) is determined by the conditions that v_n be outgoing at infinity and satisfy the boundary condition $v_n(a) = 0$, or $\frac{d}{d\xi}v_n(a) = 0$. Using standard techniques we first solve for $v_n(\xi)$ and then substitute into (14) to obtain

$$(16) \quad u(\xi,\eta) = \sum_{n=0}^{\infty} \left\{ \omega_n \int_0^{2\pi} c_n^2(\eta)\, d\eta \right\}^{-1} v_n^{(1)}(\xi_>) \left\{ v_n^{(2)}(\xi_<) - \frac{\Omega v_n^{(2)}(a)}{\Omega v_n^{(1)}(a)}\, v_n^{(1)}(\xi_<) \right\} c_n(\eta).$$

In (16), $\Omega \equiv 1$ if the boundary condition is given by (4), while $\Omega \equiv \frac{d}{d\xi}$ if the boundary condition is given by (5). The symbols $\xi_>$, $\xi_<$ mean the greater or lesser, respectively, of ξ and ξ_o. The symbol ω_n is used to denote the constant wronskian of $v_n^{(1)}$ and $v_n^{(2)}$,

$$\omega_n = v_n^{(2)} v_n^{(1)\prime} - v_n^{(1)} v_n^{(2)\prime}. \tag{17}$$

Although (16) is the exact solution to the boundary value problem, its numerical usefulness is restricted to small values of ka. For large values of ka the series is so slowly convergent that other representations of the solution must be sought. Classically, the alternative field representations have been obtained by applying the well-known Watson transformation techniques[2]. However, A.Sommerfeld[3] has shown how to obtain these representations directly without the intervening steps of the Watson transformation. His method depends upon the existence of certain expansion theorems for the special functions occurring in the solution. To apply this method to the case of the soft cylinder, we note that for certain complex values b_n^* of b outgoing solutions $v^{(1)}(\xi)$ of (9) will exist which vanish at $\xi = a$. Let us denote the value of $v^{(1)}(\xi)$ at b_n^* by $V_n^{(1)}(\xi)$ and the value of $v^{(2)}(\xi)$ at b_n^* by $V_n^{(2)}(\xi)$.† Similarly we denote $c(\eta, b_n^*)$ by $C_n(\eta)$ and $s(\eta, b_n^*)$ by $S_n(\eta)$. Then if we assume that the solution $u(\xi,\eta)$ can be written in the form

$$u(\xi,\eta) = \sum_{n=1}^{\infty} \Phi_n(\eta) V_n^{(1)}(\xi) \tag{18}$$

---

† In the case of the hard cylinder, the boundary condition is (5), and then $V_n^{(1)}(\xi)$ will be used to denote the outgoing solution of (9), whose derivative vanishes at $\xi = a$. Of course the values of b will change, depending upon the boundary condition. We will use b_n^* to denote generically either set of eigenvalues. When it becomes necessary to distinguish between the sets of eigenvalues we will use $b_n^{(1)}$ to denote eigenvalues corresponding to the boundary condition (4) and $b_n^{(2)}$ to denote eigenvalues corresponding to the boundary condition (5).

we find, upon substituting (18) into (6) and making use of the orthogonality properties of the $V_n^{(1)}(\xi)$, that $\Phi_n(\eta)$ satisfies the equation

$$(19) \qquad \Phi_n'' + k^2\left(b_n^{*2} - \cos^2\eta\right)\Phi_n = V_n^{(1)}(\xi_o)\delta(\eta)\left[\int_a^\infty \left[V_n^{(1)}(\xi)\right]^2 d\xi\right]^{-1}.$$

We now solve (19) by making use of the condition that $\Phi_n(\eta)$ must be single-valued in η and substitute into (18) to obtain in the region $0 \leq \eta \leq 2\pi$

$$(20) \qquad u(\xi,\eta) = -\frac{1}{2}\sum_{n=1}^{\infty} \frac{C_n(\eta-\pi)}{C_n'(\pi)} \frac{V_n^{(1)}(\xi_o)V_n^{(1)}(\xi)}{\int_a^\infty \left[V_n^{(1)}(\xi)\right]^2 d\xi}$$

It is convenient for later comparisons to write (20) in a different form. This is done by expressing the normalization integral in (20) as

$$(21) \qquad \int_a^\infty \left[V_n^{(1)}\right]^2 d\xi = \left(2k^2 b_n^*\right)^{-1} W_n \frac{\partial}{\partial b}\left(\Omega V_n^{(1)}(a)\right)\Big/ \Omega V_n^{(2)}(a),$$

where $b_n^* = b_n^{(1)}$ if $\Omega = 1$ or $b_n^* = b_n^{(2)}$ if $\Omega = \frac{d}{d\xi}$ and W_n is the wronskian of $V_n^{(1)}$ and $V_n^{(2)}$. Upon using (21), (20) becomes

$$(22) \qquad u(\xi,\eta) = -k^2\sum_{n=1}^{\infty} \frac{b_n^*}{W_n} \frac{C_n(\eta-\pi)}{C_n'(\pi)} V_n^{(1)}(\xi_o)V_n^{(1)}(\xi) \frac{\Omega V_n^{(2)}(a)}{\frac{\partial}{\partial b}\Omega V_n^{(1)}(a)}.$$

To obtain (21) we write the equation for $V_n^{(1)}(\xi)$ and $v^{(1)}(\xi)$, multiply the first by $v^{(1)}(\xi)$ and the second by $V_n^{(1)}(\xi)$, and subtract to obtain

$$(23) \qquad V_n^{(1)''}v^{(1)} - V_n^{(1)}v^{(1)''} - k^2(b_n^{*2}-b^2)V_n^{(1)}v^{(1)} = 0.$$

Transposing and integrating, we obtain

$$(24) \qquad \int_a^\infty V_n^{(1)}v^{(1)}d\xi = k^{-2}(b_n^{*2}-b^2)^{-1}\left[V_n^{(1)'}v^{(1)} - V_n^{(1)}v^{(1)'}\right]_a^\infty.$$

Noting that the contribution from the upper limit on the right of (24) vanishes because of the outgoing condition we obtain in the limit as $b \to b_n^*$

$$(25) \qquad \int_a^\infty \left[V_n^{(1)}\right]^2 d\xi = (2k^2 b_n^*)^{-1} \left\{ V_n^{(1)'}(a) \frac{\partial}{\partial b} V_n^{(1)}(a) - V_n^{(1)}(a) \frac{\partial}{\partial b} V_n^{(1)'}(a) \right\} .$$

Equation (21) then follows if we notice that $V_n^{(1)}(a) = 0$ if $\Omega = 1$, while $V_n^{(1)'}(a) = 0$ if $\Omega = \frac{d}{d\xi}$.

In Appendix III a rigorous derivation of (22) is presented. In that appendix it is shown that (22) is valid at all points except those for which $\eta = 0$. However, the series is too slowly convergent at all points in the geometrically lit region to be numerically useful. Thus (22), as we will show in appendix III, is of use in obtaining the asymptotic expansion only in the geometric shadow of the cylinder.

3. The asymptotic expansion of the diffracted field

In order to expand asymptotically as $k \to \infty$ the diffracted field in the shadow region as given by (22), we make use of the asymptotic expansions contained in Appendix II. When (II-12) of Appendix II is substituted into (II-13) we find

$$(26) \qquad C_n(\eta) \sim (\sinh a)^{1/2} (\cosh^2 a - \cos^2 \eta)^{-1/4} \cos\left[k \int_0^\eta (\cosh^2 a - \cos^2 \eta)^{1/2} d\eta \right.$$
$$\left. + k^{1/3} \tau_n^* (\sinh a \cosh a)^{2/3} \int_0^\eta (\cosh^2 a - \cos^2 \eta)^{-1/2} d\eta \right] .$$

Similarly, the substitution of (II-12) into (II-7) yields

$$(27)\qquad V_n^{(1)}(\xi) \sim e^{i\pi/12} k^{-1/6} 2^{-1} \pi^{-1/2} (\cosh^2\xi - \cosh^2 a)^{-1/4}$$
$$\exp\left[ik\int_a^\xi (\cosh^2\xi - \cosh^2 a)^{-1/2} d\xi - ik^{1/3}\tau_n^*(\sinh a \cosh a)^{2/3}\right.$$
$$\left.\int_a^\xi (\cosh^2\xi - \cosh^2 a)^{-1/2} d\xi\right] .$$

Also we find

$$(28)\qquad C_n'(\eta) \sim -k(\sinh a)^{1/2}(\cosh^2 a - \cos^2\eta)^{1/4}$$
$$\sin\left[k\int_0^\eta (\cosh^2 a - \cos^2\eta)^{1/2} d\eta + k^{1/3}\tau_n^*(\sinh a \cosh a)^{2/3}\right.$$
$$\left.\int_0^\eta (\cosh^2 a - \cos^2\eta)^{-1/2} d\eta\right] .$$

To complete the asymptotic expansion of (22) we need the asymptotic expansion of

$$(29)\qquad \Omega V_n^{(2)}(a)\left\{W_n \frac{\partial}{\partial b}\Omega V_n^{(1)}(a)\right\}^{-1} .$$

Let us first consider the case of the acoustically soft elliptic cylinder for which $\Omega = 1$. Then (29) may be written as

$$(30)\qquad \left\{V_n^{(1)\prime}(a)\,\frac{\partial}{\partial b}\,V_n^{(1)}(a)\right\}^{-1} .$$

To evaluate (30) we note that (II-5) may be differentiated with respect to both ξ and b to obtain

$$(31) \qquad v_n^{(1)\prime}(a) \sim \pi^{-1}\zeta_n^{1/4}3^{2/3}(2\tau_n^*)^{-1/4}k^{5/6}(\sinh a)^{-1/6}(\cosh a)^{1/12}e^{-i\pi/3}A'(q_n)\frac{\partial\zeta_n}{\partial\xi}$$

and

$$(32) \qquad \frac{\partial}{\partial b}\, v_n^{(1)}(a) \sim \pi^{-1}\zeta_n^{1/4}3^{2/3}(2\tau_n^*)^{-1/4}k^{5/6}(\sinh a)^{-1/6}(\cosh a)^{1/12}e^{-i\pi/3}A'(q_n)\frac{\partial\zeta_n}{\partial b} \,.$$

Differentiating (II-2), we find

$$(33) \qquad \frac{\partial\zeta_n}{\partial\xi} \sim -\zeta_n^{-1/2}(2\tau_n^*)^{1/2}k^{-1/3}(\sinh a)^{1/3}(\cosh a)^{-1/6}$$

and

$$(34) \qquad \frac{\partial\zeta_n}{\partial b} \sim \frac{\partial\zeta_n}{\partial\tau_n^*}\;\frac{\partial\tau_n^*}{\partial b} \,.$$

Using the above together with (II-8) and (II-12), we find that (30) becomes

$$(35) \qquad \left\{v_n^{(1)\prime}(a)\,\frac{\partial}{\partial b}\, v_n^{(1)}(a)\right\}^{-1} \sim \pi^2 e^{-i\pi/3}3^{-4/3}2^{-1/3}(\cosh a)^{-1/3}k^{-4/3}$$
$$(\sinh a)^{2/3}\left[A'(q_n)\right]^{-2} .$$

When the expansions given above are substituted into (22) we find that

$$(36) \qquad u(\xi,\eta) \sim e^{-2\pi i/3}k^{-2/3}\pi(\sinh a)^{1/6}(\cosh a)^{2/3}12^{-1}6^{-1/3}$$
$$\left(\cosh^2\xi-\cosh^2 a\right)^{-1/4}\left(\cosh^2\xi_0-\cosh^2 a\right)^{-1/4}\left(\cosh^2 a-\cos^2\eta\right)^{-1/4}$$
$$\exp\left[ik\int_a^{\xi}\left(\cosh^2\xi-\cosh^2 a\right)^{1/2}d\xi\right]\ \exp\left[ik\int_a^{\xi_0}\left(\cosh^2\xi-\cosh^2 a\right)^{1/2}d\xi\right]$$

$$\sum_{n=0}^{\infty}\Big[A'\Big(q_n^{(1)}\Big)\Big]^{-2}\bigg\{\exp\bigg[ik\int_0^{\eta}\Big(\cosh^2 a-\cos^2\eta\Big)^{1/2}d\eta$$

$$+\,ik^{1/3}\tau_n^{(1)}(\sinh a\cosh a)^{2/3}\int_0^{\eta}\Big(\cosh^2 a-\cos^2\eta\Big)^{-1/2}d\eta\bigg]$$

$$+\exp\bigg[-ik\int_0^{\eta}\Big(\cosh^2 a-\cos^2\eta\Big)^{1/2}d\eta$$

$$-\,ik^{1/3}\tau_n^{(1)}(\sinh a\cosh a)^{2/3}\int_0^{\eta}\Big(\cosh^2 a-\cos^2\eta\Big)^{-1/2}d\eta$$

$$+\,ik\int_0^{2\pi}\Big(\cosh^2 a-\cos^2\eta\Big)^{1/2}d\eta$$

$$+\,ik^{1/3}\tau_n^{(1)}(\sinh a\cosh a)^{2/3}\int_0^{2\pi}\Big(\cosh^2 a-\cos^2\eta\Big)^{-1/2}d\eta\bigg]\bigg\}$$

$$\exp\bigg[-ik^{1/3}\tau_n^{(1)}(\sinh a\cosh a)^{2/3}\bigg(\int_a^{\xi}\Big(\cosh^2\xi-\cosh^2 a\Big)^{-1/2}d\xi$$

$$+\int_a^{\xi_0}\Big(\cosh^2\xi-\cosh^2 a\Big)^{-1/2}d\xi\bigg)\bigg]$$

$$\bigg\{1-\exp\bigg[2ik\int_0^{\pi}\Big(\cosh^2 a-\cos^2\eta\Big)^{1/2}d\eta$$

$$+\,2ik^{1/3}\tau_n^{(1)}(\sinh a\cosh a)^{2/3}\int_0^{\pi}\Big(\cosh^2 a-\cos^2\eta\Big)^{-1/2}d\eta\bigg]\bigg\}^{-1}.$$

For the case of the rigid cylinder a similar result obtains. The only difference is that $\tau_n^{(1)}$ must be replaced by $\tau_n^{(2)}$ and that the first line in (36) must be replaced by

$$(37)\qquad e^{-2\pi i/3}\,k^{-2/3}\,\pi(\sinh a)^{1/6}\,(\cosh a)^{2/3}\,4^{-1}\,6^{-1/3},$$

while $\left[A'\left(q_n^{(1)}\right)\right]^{-2}$ must be replaced by $\left[q_n^{(2)}A^2\left(q_n^{(2)}\right)\right]^{-2}$.

Equation (36) is the asymptotic expansion of the diffracted field. In the next section we will construct the field geometrically and show that it is in agreement with (36).

4. The geometric construction of the diffracted field

In [1] the geometric construction of the field diffracted by any cylinder of convex cross section is described. When this construction is applied to the present case we find that the diffracted field is given by

$$(38)\quad u^d(P) = A_o\left(\rho_1 s_1\right)^{-1/2}\exp\left[ik_1\left(\rho_1+t_1+s_1\right)\right]\sum_{n=0}^{\infty}B_n(P_1)B_n(Q_1)\exp\left[i\tau_n k_1^{1/3}\int_o^{t_1} b^{-2/3}(s)ds\right]$$
$$\left\{1-\exp\left[ik_1T+i\tau_n k_1^{1/3}\int_o^{T} b^{-2/3}(s)ds\right]\right\}^{-1}$$
$$+A_o\left(\rho_2 s_2\right)^{-1/2}\exp\left[ik_1\left(\rho_2+t_2+s_2\right)\right]\sum_{n=0}^{\infty}B_n(P_2)B_n(Q_2)\exp\left[i\tau_n k_1^{1/3}\int_o^{t_2} b^{-2/3}(s)ds\right]$$
$$\left\{1-\exp\left[ik_1T+i\tau_n k_1^{1/3}\int_o^{T} b^{-2/3}(s)ds\right]\right\}^{-1}.$$

In the above ρ_1 and ρ_2 are the straight line distances from the source, $Q(\xi_o,0)$, to the cylinder along the tangents to the cylinder; s_1 and s_2 are the distances along the tangents from the cylinder to the point of observation $P(\xi,\eta)$, (see Figure 1). The points of tangency are denoted by $Q_1(a,\eta_1)$, $P_1(a,\eta_2)$, $Q_2(a,\eta_3)$, and $P_2(a,\eta_4)$.

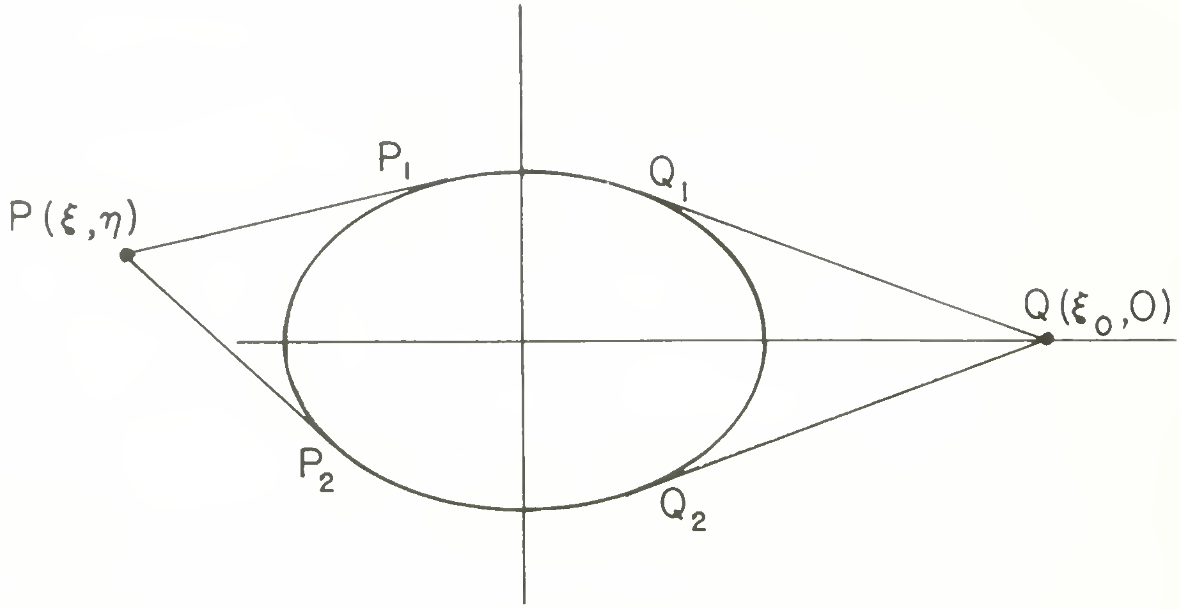

Figure 1: The two families of rays emanating from a line source located at Q and diffracted around an elliptic cylinder to a point P. The upper family is incident on the cylinder at Q_1 and leaves the cylinder tangentially at P_1, while the lower family is incident on the cylinder at Q_2 and leaves the cylinder tangentially at P_2.

The quantity t_1 is the arc length of the ellipse from Q_1 to P_1, and t_2 is the distance from Q_2 to P_2. The incident cylindrical wave at Q_1 or Q_2 is asymptotically given by $A_o\rho^{-1}e^{ik_1\rho}$. Thus

$$(39) \qquad A_o = e^{i\pi/4}(8k_1\pi)^{-1/2}.$$

The radius of curvature of the ellipse is denoted by b, and the integrals involving b are integrals with respect to arc length. Thus $\int_o^{t_1} b^{-2/3}(s)ds$ is the integral from Q_1 to P_1, while $\int_o^T b^{-2/3}(s)ds$ represents the integral taken over the entire ellipse, T being the total arc length of the ellipse. In the case of the soft cylinder the τ_n are the $\tau_n^{(1)}$ previously defined, while in the case of the rigid cylinder $\tau_n = \tau_n^{(2)}$. The diffraction coefficient B_n depends, geometrically, only upon the radius of curvative b at the point at which it is evaluated and is given in the case of the soft cylinder by

$$(40)\qquad B_n = \pi^{3/4}\, 2^{1/4}\, 6^{-2/3}\, k_1^{-1/12}\left[A'\left(q_n^{(1)}\right)\right]^{-1} e^{i\pi/24}\, b^{1/6},$$

and, in the case of the rigid cylinder, by

$$(41)\qquad B_n = \pi^{3/4}\, 2^{-1/4}\, 6^{-1/6}\, k_1^{-1/12}\left[q_n^{(2)} A\left(q_n^{(2)}\right)\right]^{-1} e^{i\pi/24}\, b^{1/6}.$$

The quantities q_n and A have been defined in Appendix II.

Let us first calculate the field at P due to the family of rays which is incident at Q_1. To do this we must compute the geometrical quantities entering into the first two lines of (38). Simple geometric calculations show that

$$(42)\qquad \eta_1 = \arccos\left(\cosh a/\cosh \xi_0\right); \qquad 0 < \eta_1 < \pi/2$$

and

$$(43)\qquad \rho_1 = h \tanh \xi_0 \left(\cosh^2\xi_0 - \cosh^2 a\right)^{1/2}.$$

Also, η_2 is the smaller solution of the equation

$$(44)\qquad \cosh \xi \sinh a \cos \eta \cos \eta_2 + \sinh \xi \cosh a \sin \eta \sin \eta_2 = \sinh a \cosh a$$

and

$$(45)\qquad s_1 = h\left\{\left(\cosh \xi \cos \eta - \cosh a \cos \eta_2\right)^2 + \left(\sinh \xi \sin \eta - \sinh a \sin \eta_2\right)^2\right\}^{1/2}.$$

Further calcualtions show that

$$(46)\qquad t_1 = h \int_{\eta_1}^{\eta_2} \left(\cosh^2 a - \cos^2 \eta\right)^{1/2} d\eta,$$

and

$$(47) \qquad T = h\int_{0}^{2\pi} (\cosh^2 a-\cos^2\eta)^{1/2} d\eta.$$

The radius of curvative as a function of the angular variable η on the ellipse is given by

$$(48) \qquad b(\eta) = h(\cosh a \sinh a)^{-1}\left\{\cosh^2 a-\cos^2\eta\right\}^{3/2}.$$

Therefore,

$$(49) \qquad \int_{0}^{t_1} b^{-2/3} ds = h^{1/3}(\cosh a \sinh a)^{2/3}\int_{\eta_1}^{\eta_2}\left(\cosh^2 a-\cos^2\eta\right)^{-1/2} d\eta,$$

and

$$(50) \qquad \int_{0}^{T} b^{-2/3} ds = h^{1/3}(\cosh a \sinh a)^{2/3}\int_{0}^{2\pi}\left(\cosh^2 a-\cos^2\eta\right)^{-1/2} d\eta.$$

Using the above expressions, we find that the field in the case of the soft cylinder due to the rays incident at Q_1 is given by

$$(51) \quad u_1^d(P) = e^{i\pi/3}k^{-2/3}\pi(12)^{-1}6^{-1/3}(\cosh a \sinh a)^{-1/3}$$

$$\left\{\cosh^2 a \sin^2\eta_1+\sinh^2 a \cos^2\eta_1\right\}^{1/4}\left\{\cosh^2 a \sin^2\eta_2+\sinh^2 a \cos^2\eta_2\right\}^{1/4}$$

$$\left(\coth \xi_o\right)^{1/2}\left(\cosh^2\xi_o-\cosh^2 a\right)^{-1/4}$$

$$\left\{\left(\cosh \xi \cos\eta-\cosh a \cos \eta_2\right)^2+\left(\sinh \xi \sin\eta- \sinh a \sin \eta_2\right)^2\right\}^{-1/4}$$

$$\exp\left[ik\left\{\tanh \xi_o\left(\cosh^2\xi_o-\cosh^2 a\right)^{1/2}\right.\right.$$

$$+\left[\left(\cosh \xi \cos\eta-\cosh a \cos \eta_2\right)^2+\left(\sinh \xi \sin\eta- \sinh a \sin \eta_2\right)^2\right]^{1/2}$$

$$\left.\left.+\int_{\eta_1}^{\eta_2}\left(\cosh^2 a-\cos^2\eta\right)^{1/2} d\eta\right\}\right]$$

$$\sum_{n=0}^{\infty} \left[A'\left(q_n^{(1)}\right)\right]^{-2} \exp\left[i\tau_n^{(1)}k^{1/3}(\cosh a \sinh a)^{2/3}\int_{\eta_1}^{\eta_2}\left(\cosh^2 a-\cos^2\eta\right)^{-1/2} d\eta\right]$$

$$\left\{1-\exp\left[ik\int_0^{2\pi}\left(\cosh^2 a-\cos^2\eta\right)^{1/2} d\eta\right.\right.$$

$$\left.\left.+i\tau_n^{(1)}k^{1/3}(\cosh a \sinh a)^{2/3}\int_0^{2\pi}\left(\cosh^2 a-\cos^2\eta\right)^{-1/2} d\eta\right]\right\}^{-1} .$$

A similar calculation holds for the field $u_2^d(P)$ due to the rays incident at Q_2. In fact, $u_2^d(P)$ has the same form as $u_1^d(P)$ except that η_1 is to be replaced by η_3, and η_2 is to be replaced by η_4. Also $\eta_3 = 2\pi-\eta_1$, while η_4 is the other solution of (44) that is, η_4 satisfies the same equation as η_2 with $0 < \eta_2 < \eta_4 < 2\pi$.

The total field is then given by

$$u^d(P) = u_1^d(P) + u_2^d(P). \tag{52}$$

Equation (51) can be further simplified by making use of (42) and (44).* Upon doing this and simplifying $u_2^d(P)$ in a similar way we find that (52) becomes

$$u^d(P) = e^{i\pi/3}k^{-2/3}\pi(12)^{-1}6^{-1/3} \tag{53}$$

$$\left(\cosh^2\xi-\cosh^2 a\right)^{-1/4}\left(\cosh^2\xi_o-\cosh^2 a\right)^{-1/4}\left(\cosh^2 a-\cos^2\eta\right)^{-1/4}$$

$$\exp\left[ik \tanh \xi_o\left(\cosh^2\xi_o-\cosh^2 a\right)^{1/2}\right]$$

---

* In Appendix I the only non-trivial calculation necessary to obtain (53) from (52) is indicated.

$$
\begin{aligned}
\sum_{n=0}^{\infty} \left[A'\left(q_n\right)\right]^{-2} \Bigg\{ \exp\Bigg[ik\Bigg\{ \Big[\left(\cosh \xi \cos \eta - \cosh a \cos \eta_2\right)^2 \\
+ \left(\sinh \xi \sin \eta - \sinh a \sin \eta_2\right)^2\Big]^{1/2} + \int_{\eta_1}^{\eta_2} \left(\cosh^2 a - \cos^2 \eta\right)^{1/2} d\eta \Bigg\} \\
+ i\tau_n^{(1)} k^{1/3} \left(\cosh a \sinh a\right)^{2/3} \int_{\eta_1}^{\eta_2} \left(\cosh^2 a - \cos^2 \eta\right)^{-1/2} d\eta \Bigg] \\
+ \exp\Bigg[ik\Bigg\{ \Big[\left(\cosh \xi \cos \eta - \cosh a \cos \eta_4\right)^2 \\
+ \left(\sinh \xi \sin \eta - \sinh a \sin \eta_4\right)^2\Big]^{1/2} \\
+ \int_{\eta_4}^{\eta_3} \left(\cosh^2 a - \cos^2 \eta\right)^{1/2} d\eta \Bigg\} \\
+ i\tau_n^{(1)} k^{1/3} \left(\cosh a \sinh a\right)^{2/3} \int_{\eta_4}^{\eta_3} \left(\cosh^2 a - \cos^2 \eta\right)^{-1/2} d\eta \Bigg] \Bigg\} \\
\Bigg\{ 1 - \exp\Bigg[ik \int_0^{2\pi} \left(\cosh^2 a - \cos^2 \eta\right)^{1/2} d\eta \\
+ i\tau_n^{(1)} k^{1/3} \left(\cosh a \sinh a\right)^{2/3} \int_0^{2\pi} \left(\cosh^2 a - \cos^2 \eta\right)^{-1/2} d\eta \Bigg] \Bigg\}^{-1} .
\end{aligned}
$$

For the rigid cylinder exactly the same modifications as mentioned after (36) must be made.

It is clear, then, comparing (36) and (53), that their amplitudes are in agreement for both the soft and rigid cylinders. For the phases to be identical we must show

$$(54)\quad \int_a^{\xi}\left(\cosh^2\xi-\cosh^2 a\right)^{1/2}d\xi+\int_a^{\xi_o}\left(\cosh^2\xi-\cosh^2 a\right)^{1/2}d\xi+\int_o^{\eta}\left(\cosh^2 a-\cos^2\eta\right)^{1/2}d\eta$$

$$=\left[\left(\cosh\xi\cos\eta-\cosh a\cos\eta_2\right)^2+\left(\sinh\xi\sin\eta-\sinh a\sin\eta_2\right)^2\right]^{1/2}$$

$$+\tanh\xi_o\left(\cosh^2\xi_o-\cosh^2 a\right)^{1/2}+\int_{\eta_1}^{\eta_2}\left(\cosh^2 a-\cos^2\eta\right)^{1/2}d\eta,$$

$$(55)\quad \int_a^{\xi}\left(\cosh^2\xi-\cosh^2 a\right)^{1/2}d\xi+\int_a^{\xi_o}\left(\cosh^2\xi-\cosh^2 a\right)^{1/2}d\xi-\int_o^{\eta}\left(\cosh^2 a-\cos^2\eta\right)^{1/2}d\eta$$

$$+\int_o^{2\pi}\left(\cosh^2 a-\cos^2\eta\right)^{1/2}$$

$$=\left[\left(\cosh\xi\cos\eta-\cosh a\cos\eta_4\right)^2+\left(\sinh\xi\sin\eta-\sinh a\sin\eta_4\right)^2\right]^{1/2}$$

$$+\tanh\xi_o\left(\cosh^2\xi_o-\cosh^2 a\right)^{1/2}+\int_{\eta_4}^{\eta_3}\left(\cosh^2 a-\cos^2\eta\right)^{1/2}d\eta,$$

$$(56)\quad -\int_a^{\xi}\left(\cosh^2\xi-\cosh^2 a\right)^{-1/2}d\xi-\int_a^{\xi_o}\left(\cosh^2\xi-\cosh^2 a\right)^{-1/2}d\xi+\int_o^{\eta}\left(\cosh^2 a-\cos^2\eta\right)^{-1/2}d\eta$$

$$=\int_{\eta_1}^{\eta_2}\left(\cosh^2 a-\cos^2\eta\right)^{-1/2}d\eta,$$

and

$$(57)\quad -\int_a^{\xi}\left(\cosh^2\xi-\cosh^2 a\right)^{-1/2}d\xi-\int_a^{\xi_o}\left(\cosh^2\xi-\cosh^2 a\right)^{-1/2}d\xi$$

$$-\int_o^{\eta}\left(\cosh^2 a-\cos^2\eta\right)^{-1/2}d\eta+\int_o^{2\pi}\left(\cosh^2 a-\cos^2\eta\right)^{-1/2}d\eta$$

$$=\int_{\eta_4}^{\eta_3}\left(\cosh^2 a-\cos^2\eta\right)^{-1/2}d\eta.$$

In Appendix I these identities will be proved, showing that the geometric construction is in agreement with the asymptotic expansion of the exact solution.

Acknowledgement

The author wishes to express his appreciation to Professor Joseph B. Keller for suggesting the problem considered in this paper and for giving freely of his time in aiding and guiding the research.

Comparison of the asymptotic expansion with the geometric construction

As we saw in Section 4 the asymptotic expansion of the exact solution is in agreement with the geometric result if (54)-(57) can be proved and (53) can be shown to follow from (52). If in (51) and the similar expression for $u_2^d(P)$ we substitute (42) and the similar expression for η_3 we find that (53) follows if the following identity holds:

$$\text{(I-1)}\quad \frac{\cosh^2 a \sin^2\eta_2+\sinh^2 a \cos^2\eta_2}{(\cosh \xi \cos \eta-\cosh a \cos \eta_2)^2+(\sinh \xi \sin \eta-\sinh a \sin \eta_2)^2}$$

$$= \frac{\sinh^2 a \cosh^2 a}{(\cosh^2 a-\cos^2\eta)(\cosh^2\xi-\cosh^2 a)} .$$

Upon carrying out the squaring in the denominator, using (44) to express the cross product term $\sinh \xi \sin \eta \sinh a \sin \eta_2$ as

$$\sinh^2 a - (\cosh a)^{-1}\cosh \xi \sinh^2 a \cos \eta \cos \eta_2 ,$$

and finally expressing the left-hand side of (I-1) in terms of cosines and hyperbolic cosines, we find that it becomes

$$\text{(I-2)}\quad \frac{\cosh^2 a - \cos^2\eta_2}{\cos^2\eta + \cosh^2\xi - \cosh^2 a + \cos^2\eta_2 - 2(\cosh a)^{-1}\cosh \xi \cos \eta \cos \eta_2}$$

Now, by transposing the first term of (44) to the right-hand side and squaring, we find that

$$\text{(I-3)}\quad -2(\cosh a)^{-1}\cosh \xi \cos \eta \cos \eta_2$$

$$= (\sinh a \cosh a)^{-2}\Big\{\sinh^2\xi \cosh^2 a \sin^2\eta \sin^2\eta_2$$

$$- \cosh^2\xi \sinh^2 a \cos^2\eta \cos^2\eta_2 - \sinh^2 a \cosh^2 a\Big\} .$$

When (I-3) is substituted into the denominator of (I-2) we find, after some manipulation, that the denominator may be written as

$$\text{(I-4)}\qquad (\sinh a \cosh a)^{-2}\left(\cosh^2 a-\cos^2\eta_2\right)\left(\cosh^2\xi-\cosh^2 a\right)\left(\cosh^2 a-\cos^2\eta\right).$$

Upon substituting (I-4) into (I-2) we find that (I-1) holds.

The identities (54)-(57) are based on addition formulas for elliptic integrals. Accordingly, we introduce the elliptic integrals of the first and second kind defined by

$$\text{(I-5)}\qquad F(\phi,\kappa) = \int_0^{\phi}\left(1-\kappa^2\sin^2 t\right)^{-1/2}dt$$

and

$$\text{(I-6)}\qquad E(\phi,\kappa) = \int_0^{\phi}\left(1-\kappa^2\sin^2 t\right)^{1/2}dt\ .$$

As a first step in verifying (54) we show

$$\text{(I-7)}\qquad \int_a^{\xi_0}\left(\cosh^2\xi-\cosh^2 a\right)^{1/2}d\xi + \int_0^{\eta_1}\left(\cosh^2 a-\cos^2\eta\right)^{1/2}d\eta$$

$$= \tanh \xi_0 \left(\cosh^2\xi_0 - \cosh^2 a\right)^{1/2}.$$

In terms of the E function we may write this as

$$\text{(I-8)}\quad E(ia,\kappa)-E(i\xi_0,\kappa)+E(\eta_1,\kappa) = \tanh \xi_0 (\sinh a)^{-1}\left\{\cosh^2\xi_0-\cosh^2 a\right\}^{1/2},$$

with

$$\text{(I-9)}\qquad \kappa = (i \sinh a)^{-1}.$$

To verify (I-8) we make use of the addition theorem for the elliptic functions of the second kind[4] which states that

$$\text{(I-10)}\qquad E(\chi) = E(\phi) + E(\Psi) - \kappa^2\sin\phi\ \sin\Psi\ \sin\chi$$

if χ, Ψ, and ϕ are related by

$$\text{(I-11)} \qquad \left(1-\kappa^2\sin^2\phi\,\sin^2\psi\right)\sin\chi = \sin\phi\,\cos\psi\,\left(1-\kappa^2\sin^2\psi\right)^{1/2} + \sin\psi\,\cos\phi\,\left(1-\kappa^2\sin^2\phi\right)^{1/2},$$

and

$$\text{(I-12)} \qquad \left(1-\kappa^2\sin^2\phi\,\sin^2\psi\right)\cos\chi = \cos\phi\,\cos\psi - \sin\phi\,\sin\psi\,\left(1-\kappa^2\sin^2\phi\right)^{1/2}\left(1-\kappa^2\sin^2\psi\right)^{1/2}.$$

In (I-10) the congruence sign means that the right and left-hand sides differ by a constant linear combination of the periods of the E function. When we apply (I-10) to (I-8) by choosing $\psi = \eta_1$, $\phi = ia$, and $\chi = i\xi_o$ we see first by setting $\xi_o = 0$ that the congruence must be an equality and then that (I-8) follows. The verification of (I-11) and (I-12) follows from (42) by a simple calculation.

To complete the verification of (54) we next show that

$$\text{(I-13)} \qquad \int_a^{\xi}\left(\cosh^2\xi-\cosh^2 a\right)^{1/2}d\xi + \int_{\eta_2}^{\eta}\left(\cosh^2 a-\cos^2\eta\right)^{1/2}d\eta$$

$$= \left[\left(\cosh\xi\,\cos\eta - \cosh a\,\cos\eta_2\right)^2 + \left(\sinh\xi\,\sin\eta - \sinh a\,\sin\eta_2\right)^2\right]^{1/2}.$$

The proof of (I-13) is obtained by two applications of the addition formula. We first introduce ξ^* defined by[†]

$$\text{(I-14)} \qquad \eta_2 = \pi - \arccos\frac{\cosh a}{\cosh\xi^*},$$

and break up the integral from a to ξ into two integrals: one from a to ξ^*, the other from ξ^* to ξ. Similarly, the integral from η_2 to η we write as an integral from η_2 to π plus an integral from π to η. Then in terms of elliptic integrals (I-13) may be written as

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[†] We assume here that $\pi/2 < \eta_2 < \pi$. For other ranges of η_2 similar arguments will apply to show the equality of the phases.

$$\text{(I-15)}\quad E\left(\arccos\left\{\frac{\cosh a}{\cosh \xi^*}\right\}\right) - E(i\xi^*) + E(ia) + E(\eta-\pi) - E(i\xi) + E(i\xi^*)$$

$$= (\sinh a)^{-1}\left[\left(\cosh \xi \cos \eta - \cosh a \cos \eta_2\right)^2\right.$$

$$\left. + \left(\sinh \xi \sin \eta - \sinh a \sin \eta_2\right)^2\right]^{1/2}.$$

In the above, as in the following, we suppress the modulus, (III-9), of the elliptic integrals. Now upon comparing the first three terms of (I-15) with (I-8) we see that

$$\text{(I-16)}\quad E\left(\arccos\left\{\frac{\cosh a}{\cosh \xi^*}\right\}\right) - E(i\xi^*) + E(ia) = \tanh \xi^* (\sinh a)^{-1}\left\{\cosh^2\xi^* - \cosh^2 a\right\}^{1/2}.$$

Upon applying the addition formula (I-10) to the remaining terms in (I-15) we find

$$\text{(I-17)}\qquad E(\eta-\pi) - E(i\xi) + E(i\xi^*) = -(\sinh a)^{-2}\sin \eta \sinh \xi^* \sinh \xi.$$

Again (I-11) and (I-12) may be verified with $\phi = \eta-\pi$, $\Psi = i\xi^*$, and $\lambda = i\xi$.

To prove (I-15) we first observe that by (I-14) the points (a,η_2), (ξ,η), and (ξ^*,π) are collinear. Expressing the fact that the distance between (a,η_2) and (ξ,η) is equal to the distance between (a,η_2) and (ξ^*,π) minus the distance between (ξ,η) and (ξ^*,π) we find that the right-hand side of (I-15) can be written as

$$\text{(I-18)}\quad [\sinh a]^{-1}\left\{\left(\cosh \xi^* + \cosh a \cos \eta_2\right)^2 + \sinh^2 a \sin^2 \eta_2\right\}^{1/2}$$

$$- [\sinh a]^{-1}\left\{\left(\cosh \xi^* + \cosh \xi \cos \eta\right)^2 + \sinh^2 \xi \sin^2 \eta\right\}^{1/2}.$$

Now, upon using (I-14) in the first line of (I-18) we see that this first line is in agreement with (I-16). To show the agreement of the second line with (I-17) we first substitute the values of cos η_2 and sin η_2 in terms of ξ^* into (44) obtaining

$$\text{(I-19)} \qquad \cosh \xi^* + \cosh \xi \cos \eta = -(\sinh a)^{-1} \sinh \xi \sin \eta \left(\cosh^2 \xi^* - \cosh^2 a\right)^{1/2} .$$

When (I-19) is substituted into the second line of (I-18) the agreement with (I-17) follows readily. This completes the verification of (I-13) and hence of (54). Similar arguments may be used to verify (55).

The verification of (56) and (57) follow from (54) and (55) by use of the addition formula for the elliptic integrals of the first kind[5] which states that

$$\text{(I-20)} \qquad F(\alpha) \equiv F(\phi) + F(\psi) ,$$

if α, ψ, and ϕ are related by (I-11) and (I-12). Upon writing (56) and (57) in terms of the F functions we find that they agree with (54) and (55), respectively, if we change the E functions to F functions and drop the inhomogeneous terms. Since (I-20) differs from (I-10) only in being homogeneous, (56) and (57) follow immediately.

Appendix II

Asymptotic expansions of the elliptic cylinder functions

To obtain the asymptotic expansions of the solutions of (9) we apply the methods of F.W.J. Olver[6]. Olver gives asymptotic expansions of the solutions of the equation

$$\text{(II-1)} \qquad \frac{d^2W}{d\zeta^2} = \left(k^2\zeta + f(\zeta)\right)W$$

for large real k, uniform in the complex variable ζ. To reduce (9) to the canonical form (II-1) we introduce new variables through the equations

$$\text{(II-2)} \qquad \frac{2}{3}\zeta^{3/2} = \int_{\cosh^{-1}b}^{\xi}\left(b^2-\cosh^2x\right)^{1/2}dx$$

and

$$\text{(II-3)} \qquad W = \left(\frac{d\xi}{d\zeta}\right)^{-1/2}Y.$$

Then (9) assumes the form (II-1) with

$$\text{(II-4)} \quad f(\zeta) = \frac{1}{4}\zeta^{-2}\left(5/4-5\zeta^2\cosh^2\xi\,\sinh^2\xi\left(b^2-\cosh^2\xi\right)^{-3}-2\zeta^2\cosh 2\xi\left(b^2-\cosh^2\xi\right)^{-2}\right).$$

According to Olver the asymptotic expansion of a solution of (II-1) in any domain of the complex z plane is made up of linear combinations of appropriately chosen Airy functions. In particular the outgoing solution of (9) is given asymptotically by

$$\text{(II-5)} \qquad v^{(1)}(\xi) \sim \zeta^{1/4}\,3^{1/3}\,\pi^{-1}\left(b^2-\cosh^2\xi\right)^{-1/4}\,A\left(3^{1/3}\,e^{-i\pi/3}\,k^{2/3}\,\zeta\right).$$

In (II-5) the Airy function A(t) is defined by

$$\text{(II-6)} \qquad A(t) = \int_0^\infty \cos\left(z^3-tz\right)dz.$$

In the case when $\xi > \cosh^{-1}b$ we can expand the Airy function in (II-5) to obtain

$$\text{(II-7)} \quad v^{(1)}(\xi) \sim e^{i\pi/12}k^{-1/6}2^{-1}\pi^{-1/2}\left(\cosh^2\xi-b^2\right)^{-1/4}\exp\left[ik\int_{\cosh^{-1}b}^{\xi}\left(\cosh^2\xi-b^2\right)^{1/2}d\xi\right].$$

We can also use (II-5) to evaluate the complex eigenvalues b_n^*. For the case of the soft cylinder the eigenvalue is defined by $v^{(1)}(a) = 0$. Thus if $q_n^{(1)}$ is a zero of the Airy function we introduce $\zeta_n = \zeta(a)$ by

$$\text{(II-8)} \qquad \zeta_n = e^{i\pi/3} k^{-2/3} 3^{-1/3} q_n^{(1)},$$

and then from (2) we obtain, as the equation defining the eigenvalue,

$$\text{(II-9)} \qquad \tfrac{2}{3}\zeta_n^{3/2} = \int_{\cosh^{-1} b_n^{(1)}}^{a} \left(b_n^{(1)^2} - \cosh^2 \xi \right) d\xi .$$

Now since $\zeta_n^{3/2} = \mathcal{O}(k^{-1})$ we see that for large k, $b_n^{(1)}$ must tend to cosh a. Thus we can set $b_n^{(1)} = \cosh a + \alpha k^{-\beta}$ and determine α and β by an asymptotic comparison of the right and left-hand sides of (II-9). In this way we find

$$\text{(II-10)} \qquad b_n^{(1)} = \cosh a + e^{i\pi/3} q_n^{(1)} 6^{-1/3} k^{-2/3} (\sinh a)^{2/3} (\cosh a)^{-1/3}.$$

Exactly the same arguments hold for the case of the hard cylinder. In fact if we replace $q_n^{(1)}$ by $q_n^{(2)}$, the zeroes of $A'(q)$, we obtain the expression for $b_n^{(2)}$. Thus if we introduce

$$\text{(II-11)} \qquad \frac{e^{i\pi/3} q_n^*}{6^{1/3}} = \tau_n^*$$

we can write generically

$$\text{(II-12)} \qquad b_n^* = \cosh a + \tau_n^* k^{-2/3} (\sinh a)^{2/3} (\cosh a)^{-1/3}.$$

As before, for the soft cylinder the starred quantities are to have superscripts (1) and for the hard cylinder superscripts (2).

The asymptotic expansion as $k \to \infty$ of the solutions of (8) are obtained by using W.K.B. theory. In the range $0 \leq \eta \leq 2\pi$ we obtain

$$\text{(II-13)} \quad c(\eta,b) \sim \cos\left[k \int_0^{\eta} \left(b^2 - \cos^2\eta\right)^{1/2} d\eta \right] \left\{ \left(b^2-1\right) / \left(b^2 - \cos^2\eta\right) \right\}^{1/4}.$$

Appendix III

The alternative field representation and its convergence properties

Equation (22) of Section 2 was derived in a purely formal way under the assumption that the functions $V_n^{(1)}(\xi)$ were complete, that is, the solution $u(\xi,\eta)$ could be represented by a convergent sum of the form (18). Equation (22) can also be derived in a rigorous way from the series solution (16). To do this we employ the well-known techniques first used by Watson in the analysis of diffraction by the earth. We first express (16) as a contour integral in the complex b plane and then find that (22) is a residue evaluation of the integral. In this appendix we carry out the Watson transformation for the elliptic cylinder and then investigate the convergence properties of (22). We will find that (22) is a convergent representation of the field in all directions except $\eta = 0$. The convergence properties of (36) of Section 3 will also be examined and we will show that (36) is convergent at all points (ξ,η) in the geometric shadow.

Letting Γ be a contour which encircles the real axis (See Figure 2) we may write the sum (22) as

$$\text{(III-1)} \qquad u(\xi,\eta) = \frac{1}{2\pi i}\int_{\Gamma} f(b)\,db \, ,$$

where the function f(b) is an analytic function of b having poles at the eigenvalues b_n, and where the sum of the residues of the integral at the points b_n is given by (22). The first step in obtaining the representation (III-1) then is to find an analytic function of b vanishing at the eigenvalues b_n. To construct this function we observe that $c(-\pi,b)$ is a solution of (8). Thus we may write

$$\text{(III-2)} \qquad c(\eta-\pi,b) = \alpha(b)c(\eta,b) + \gamma(b)s(\eta,b) \, .$$

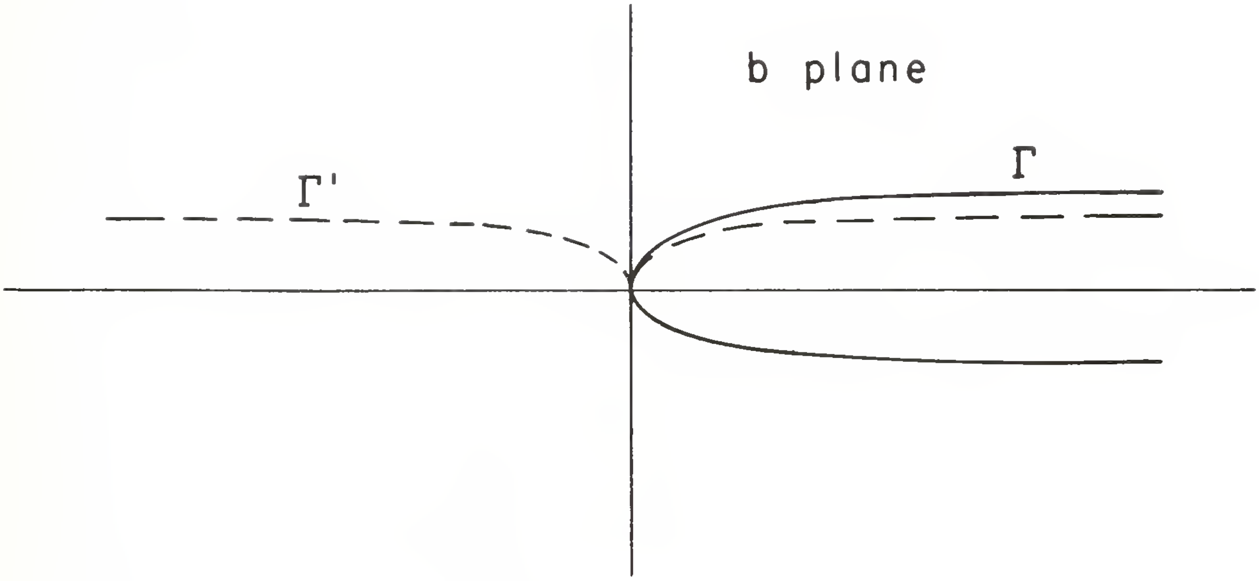

Figure 2: The paths of integration for the integral representation of the field diffracted by an elliptic cylinder. The contour Γ encircles the positive real axis, while Γ' coincides with the upper branch of Γ for Re $b \geq 0$ and is as shown for Re $b \leq 0$.

Now, upon replacing η by $-\eta$ and b by b_n, we see that $\gamma(b_n)$ must be zero. Then taking the derivative of (III-2) and setting $b = b_n$, we see that $c'(\pi, b_n) = 0$. Hence if we take for $f(b)$ in (III-1) an entire function of b divided by $c'(\pi, b)$ we will produce the correct poles for a residue evaluation.

The zeroes of $c'(\pi, b)$ are simple so that the residue evaluation of (III-1) will necessitate evaluating

$$\text{(III-3)} \qquad \left[\frac{\partial}{\partial b} c'(\pi, b) \right]_{b=b_n}$$

To evaluate (III-3) we multiply the equation satisfied by $c(\eta, b_n) = c_n(\eta)$

by $c(\eta,b)$ and multiply the equation satisfied by $c(\eta,b)$ by $c_n(\eta)$ and subtract the resulting equations to obtain

$$(\text{III-4}) \qquad c''(\eta,b)c_n(\eta) - c_n''(\eta)c(\eta,b) = k^2\left\{b^2-b_n^2\right\} cc_n \; .$$

Now, upon integrating (III-4) between $\eta = 0$ and $\eta = \pi$, we find

$$(\text{III-5}) \qquad c_n(\pi)c'(\pi,b) = -k^2\left(b^2-b_n^2\right)\int_0^\pi c(\eta,b)c_n(\eta)d\eta \; .$$

Then, dividing (III-5) by $b-b_n$ and letting $b \rightarrow b_n$, we obtain

$$(\text{III-6}) \qquad c_n(\pi)\,\frac{\partial}{\partial b}\,c'(\pi,b_n) = -2k^2 b_n \int_0^\pi c_n^2(\eta)d\eta \; .$$

We now use (III-2) to extend the range of integration in (III-6) to 2π. We observed that for $b = b_n$ the function $c_n(\eta-\pi)$ is proportional to $c_n(\eta)$. Setting $\eta = 0$ in (III-2) we obtain $\alpha(b_n) = c_n(\pi)$. Thus

$$(\text{III-7}) \qquad c_n(\eta-\pi) = c_n(\pi)c_n(\eta) \; .$$

Now upon replacing η by $\eta-\pi$ we conclude that $c_n^2(\pi) = 1$. Hence

$$(\text{III-8}) \qquad \int_\pi^{2\pi} c_n^2(\eta)d\eta = \int_\pi^{2\pi} c_n^2(\eta-\pi)d\eta = \int_0^\pi c_n^2(\eta)d\eta \; .$$

Thus we may write (III-6) as

$$(\text{III-9}) \qquad \frac{\partial}{\partial b}\,c'(\pi,b_n) = -\frac{k^2 b_n}{c_n(\pi)}\int_0^{2\pi} c_n^2(\eta)d\eta \; .$$

Finally, using (III-7) and (III-9), we see that $f(b)$ may be expressed as

$$(\text{III-10}) \qquad f(b) = -\frac{bk^2 c(\eta-\pi,b)}{\omega(b)c'(\pi,b)}\,v^{(1)}(\xi_>,b) \left\{\frac{v^{(2)}(\xi_<,b)\Omega v^{(1)}(a,b)-v^{(1)}(\xi_<,b)\Omega v^{(2)}(a,b)}{\Omega v^{(1)}(a,b)}\right\} .$$

In the integral (III-1) we now replace b by -b on that part of the contour for which Im $b<0$. Noting that $v^{(1)}(\xi,-b)$ is proportional to $v^{(1)}(\xi,b)$ and similarly that $v^{(2)}(\xi,-b)$ is proportional to $v^{(2)}(\xi,b)$, while $c(\eta,v)$ is an even function of b, we may transform (III-1) to an integral over the contour Γ' shown in Figure 2. Finally, if the integral over an infinite semi-circle in the upper half-plane vanishes, then $u(\xi,\eta)$ may be evaluated in terms of its residues in the upper half-plane. This residue evaluation then yields the representation (22).

In order to estimate the integral over a large semi-circle in the upper half-plane, asymptotic expansions as $b\to\infty$ of the functions $v(\xi)$ and $c(\eta)$ are necessary. Since such expansions are not readily available we leave the proof incomplete and show instead that the residue series converges at all points except those for which $\eta = 0$.

The first observation we make is that in the neighborhood of the eigenvalues b_n^* the function $v^{(1)}(\xi)$ must be oscillatory in character. Thus if we expand the Airy function in (II-5) the expansion must be in terms of a trigonometric rather than a pure exponential function. In what follows convergence will be shown to be exponential in character. Thus we will suppress all multiplicative algebraic factors and use a generic f to denote them. When we expand the Airy function in (II-5) we find

(III-11) $$v^{(1)}(\xi) \sim f \cos(\omega-\pi/4)$$

where

(III-12) $$\omega = \tfrac{2}{3}e^{-i\pi/2}\, k\zeta^{3/2},$$

with ζ given by (II-2). Now we argue that if we set $b = te^{i\Upsilon}$ then as $t\to\infty$ the imaginary part of ω must be zero for (III-11) to be oscillatory. Thus the real part of the integral (II-2) must vanish. This condition then yields an asymptotic expansion as $t\to\infty$ of Υ as a function of t, and thus paramaterizes the curve on which the eigenvalues lie.

In order to asymptotically expand the integral in (II-2) as $b \to \infty$ we first change the variable of integration by setting $z = \cosh x$. Then the integral becomes

$$\text{(III-13)} \qquad I = \int_b^{\cosh \xi} \left(b^2-z^2\right)^{1/2} \left(z^2-1\right)^{-1/2} dz.$$

We introduce $\delta(b)$ such that $\delta/b \to 0$ and $\delta \to \infty$ as $b \to \infty$, and then write I as the sum I_1+I_2, where I_1 is the integral from b to δ, and I_2 the integral from δ to $\cosh \xi$. Then in I_1, z is large and we may write

$$\text{(III-14)} \qquad I_1 \sim \int_b^{\delta} z^{-1}\left(b^2-z^2\right)^{1/2} dz.$$

In I_2 we first factor out b and then observe that b/z is small to obtain

$$\text{(III-15)} \qquad I_2 \sim b\int_{\delta}^{\cosh \xi} \left(z^2-1\right)^{1/2} dz.$$

The integrations for I_1 and I_2 can be carried out explicitly. When this is done and the sum $I_1 + I_2$ is formed and expanded we find that

$$\text{(III-16)} \qquad I \sim b\left\{1 - \log \frac{4b}{\sinh \xi + \cosh \xi}\right\}.$$

A more careful analysis of I shows that (III-16) is exact to within terms $O(\delta^{-1})$. Substituting $b = te^{i\gamma}$ we obtain for the real part of I

$$\text{(III-17)} \quad \operatorname{Re} I = t\left\{\cos \gamma \left[1-\log 4t\Big/(\sinh \xi + \cosh \xi)\right] + \gamma \sin \gamma\right\}.$$

Equating (III-17) to zero with $\xi = a$ we obtain as a parametric representation of the curve on which the eigenvalues lie

$$\text{(III-18)} \qquad \gamma_0 \sim \frac{\pi}{2}\left(1- \log\left\{\frac{4t}{\sinh a + \cosh a}\right\}^{-1}\right).$$

Using the above in (III-11), we find as $n \to \infty$

$$\text{(III-19)}\quad V_n^{(1)}(\xi) \sim f \exp\left[\pi t \log\left(\frac{\sinh \xi + \cosh \xi}{\sinh a + \cosh a}\right) \Big/ 2 \log \frac{4t}{\sinh a + \cosh a}\right].$$

Since the function $V_n^{(1)}(a)$ is oscillatory we see that the convergence of (22) is determined by the behavior as $n \to \infty$ of the quotient

$$\text{(III-20)}\qquad C_n(\eta-\pi)V_n^{(1)}(\xi)V_n^{(1)}(\xi_o) \Big/ C_n'(\pi).$$

Using the fact that as $n \to \infty$

$$\text{(III-21)}\qquad C_n(\eta) \sim e^{t\eta},$$

together with (III-19), we see that as $n \to \infty$ (III-20) is asymptotic to

$$\text{(III-22)}\quad f \exp\left[t\left\{|\eta-\pi| - \pi + \frac{\pi}{2} \log\left(\frac{(\sinh \xi + \cosh \xi)(\sinh \xi_o + \cosh \xi_o)}{(\sinh a + \cosh a)^2}\right)\left(2 \log \frac{4t}{\sinh a + \cosh a}\right)^{-1}\right\}\right].$$

Consequently, (22) is convergent in all directions except $\eta = 0$.

Although (22) is convergent in all directions except $\eta = 0$, it is clear from (III-21) that the series is most rapidly convergent in the direction $\eta = \pi$ and has a slower rate of convergence as η tends to 0. To be practically useful, however, the earlier terms of the series must be dominant. Thus the terms of (22) for which the b_n are given by (II-10) must be dominant; hence the terms in the sum (26) must be exponentially decaying. Since the τ_n have a positive imaginary part this requires that the expressions on the left of (56) and (57) must be positive. In Appendix I we have proved the equalities of (56) and (57). Thus for the left-hand side of (56) and (57) to be positive we must have $\eta_2 > \eta_1$ and $\eta_3 > \eta_4$. A glance at Figure 1 shows that this is just the condition that the point (ξ,η) lies in the geometric shadow of the cylinder. Hence, although (22) is convergent at points in the illuminated region of the cylinder, it is practically useful only in the shadow. In the lit region a different representation of the field which separates out the reflected field is necessary in order to obtain the asymptotic expansion of the diffracted field.

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